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Similarity invariants for a class of nilpotent operators

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In this note, all Hilbert spaces will be understood to be *complex*. If $\mathfrak{H}$ is a Hilbert space, we denote by $\mathfrak{L}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. If A belongs to $\mathfrak{L}(\mathfrak{H})$ and there is a positive integer n such that $A^n=0$ and $A^{n-1}\neq 0$, then we say A is a *nilpotent* operator of order n. If n is a positive integer, then the nilpotent operator acting on the direct sum of n copies of $\mathfrak{H}$ and defined by the $n\times n$ matrix $[A_{ij}]$ $(i,j=1,\ldots,n)$, where

$$A_{i,i+1} = 1_{\mathfrak{H}} \quad \text{for} \quad i = 1, \ldots, n-1 \quad \text{and} \quad A_{i,j} = 0_{\mathfrak{H}} \quad \text{for all other entries,}$$

is called a *Jordan block* operator of order n. (By definition, $0_{\mathfrak{H}}$, the zero operator on $\mathfrak{H}$, is a Jordan block operator or order one.) Let m be a positive integer. Suppose $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$ are Hilbert spaces and $n_1, \ldots, n_m$ are positive integers. Let $\tilde{\mathfrak{H}}_k$ be the direct sum of n_k copies of $\mathfrak{H}_k$ and J_k be the Jordan block operator of order n_k acting on $\tilde{\mathfrak{H}}_k$, $k=1, 2, \ldots, m$. An operator of the form $J_1\oplus\ldots\oplus J_m$ acting on $\tilde{\mathfrak{H}}_1\oplus\ldots\oplus\tilde{\mathfrak{H}}_m$ is called a *Jordan operator*.

Recall that if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are Hilbert spaces and X: $\mathfrak{K}_1\to\mathfrak{K}_2$ is a bounded linear transformation such that kernel $X=$kernel $X^*=\{0\}$, then X is called a *quasiaffinity*. If $A_1\in\mathfrak{L}(\mathfrak{K}_1)$, $A_2\in\mathfrak{L}(\mathfrak{K}_2)$, and there exists a quasiaffinity X: $\mathfrak{K}_1\to\mathfrak{K}_2$ such that $XA_1=A_2X$, then we say A_1 is a *quasiaffine transform* of A_2. If A_1 and A_2 are quasiaffine transforms of each other, i.e., if there exist quasiaffinities X: $\mathfrak{K}_1\to\mathfrak{K}_2$ and Y: $\mathfrak{K}_2\to\mathfrak{K}_1$ such that $XA_1=A_2X$ and $YA_2=A_1Y$, then A_1 and A_2 are said to be *quasisimilar*. Recall also that if there exists an invertible bounded linear transformation Z: $\mathfrak{K}_1\to\mathfrak{K}_2$ such that $ZA_1=A_2Z$, then A_1 and A_2 are said to be *similar*.

It is a well-known theorem of linear algebra that every nilpotent operator on a finite dimensional Hilbert space is similar to a Jordan operator. Since every Jordan operator clearly has closed range, one cannot expect this theorem to be true on an

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infinite dimensional Hilbert space, but APOSTOL, DOUGLAS, and FOIAŞ [1] recently proved that the following weakened version of the theorem is valid on any Hilbert space.

Theorem 1. *Every nilpotent operator on a Hilbert space of arbitrary dimension is quasisimilar to a Jordan operator.*

The purpose of this note is two-fold. In the first place, we present below a proof of Theorem 1 that is somewhat simpler than the argument in [1]. Secondly, essentially the same proof establishes the following result.

Theorem 2. *A nilpotent operator T on a Hilbert space is similar to a Jordan operator if and only if the range of T^k is closed, $k=1, 2, \ldots$.*

It will be convenient to use the following notation. If $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are Hilbert spaces, A belongs to $\mathfrak{L}(\mathfrak{K}_1)$, and B: $\mathfrak{K}_2 \to \mathfrak{K}_1$ is a bounded linear transformation, then we let $M(A, B)$ denote the operator

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

in $\mathfrak{L}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)$. If A: $\mathfrak{K}_1 \to \mathfrak{K}_2$ is a bounded linear transformation, then we denote by $\mathfrak{K}(A)$ the kernel of A and by $\mathfrak{R}(A)$ the range of A.

We begin with the following lemma.

Lemma 1. *Suppose J is a Jordan operator acting on a Hilbert space $\mathfrak{H}$, and suppose there are a Hilbert space $\mathfrak{K}$ and an isometry V: $\mathfrak{K} \to \mathfrak{H}$ such that $\mathfrak{R}(V) = \mathfrak{H} \ominus \mathfrak{R}(J)$. Then the operator $M(J, V)$ in $\mathfrak{L}(\mathfrak{H} \oplus \mathfrak{K})$ is unitarily equivalent to a Jordan operator.*

Proof. To say that J is a Jordan operator on $\mathfrak{H}$ means that there exist Hilbert spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$ and positive integers $n_1, \ldots, n_m$ such that if we let $\mathfrak{H}_k^{\sim}$ be the direct sum of n_k copies of $\mathfrak{H}_k$ and J_k be the Jordan block operator of order n_k on $\mathfrak{H}_k^{\sim}$ $(k=1, 2, \ldots, m)$, then $\mathfrak{H} = \mathfrak{H}_1^{\sim} \oplus \ldots \oplus \mathfrak{H}_m^{\sim}$ and $J = J_1 \oplus \ldots \oplus J_m$. Let $\mathfrak{H}_k^{\wedge} = \mathfrak{H}_k^{\sim} \ominus \mathfrak{R}(J_k)$, i.e. $\mathfrak{H}_k^{\wedge} = 0 \oplus \ldots \oplus 0 \oplus \mathfrak{H}_k$ $(k=1, 2, \ldots, m)$. It is easy to verify that $\mathfrak{R}(V) = \mathfrak{H} \ominus \mathfrak{R}(J) = \mathfrak{H}_1^{\wedge} \oplus \ldots \oplus \mathfrak{H}_m^{\wedge}$. Let U_k be the natural Hilbert space isomorphism of $\mathfrak{H}_k$ onto $\mathfrak{H}_k^{\wedge}$. Define W_k: $\mathfrak{H}_k \to \mathfrak{H}_k^{\sim}$ by setting $W_k x = U_k x$ for each x in $\mathfrak{H}_k$. Let $U = U_1 \oplus \ldots \oplus U_m$ and $W = W_1 \oplus \ldots \oplus W_m$. Define V_0: $\mathfrak{K} \to \mathfrak{R}(V)$ by setting $V_0 x = Vx$ for each $x \in \mathfrak{K}$. The linear transformations U: $\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m \to \mathfrak{R}(V)$ and V_0: $\mathfrak{K} \to \mathfrak{R}(V)$ are unitary. Hence the linear transformation

$$1_{\mathfrak{H}} \oplus U^* V_0 : \mathfrak{H} \oplus \mathfrak{K} \to \mathfrak{H} \oplus (\mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_m)$$

is unitary and

$$(1_{\mathfrak{H}} \oplus U^* V_0) M(J, V) (1_{\mathfrak{H}} \oplus U^* V_0)^* = M(J, V V_0^* U) = M(J, W).$$

Furthermore, the operator

$$M(J,W) \quad \text{in} \quad \mathfrak{L}(\mathfrak{H}\oplus(\mathfrak{H}_1\oplus\ldots\oplus\mathfrak{H}_m))$$

is unitarily equivalent to the operator

$$M(J_1,W_1)\oplus\ldots\oplus M(J_m,W_m) \quad \text{in} \quad \mathfrak{L}((\tilde{\mathfrak{H}}_1\oplus\mathfrak{H}_1)\oplus\ldots\oplus(\tilde{\mathfrak{H}}_m\oplus\mathfrak{H}_m)).$$

Thus, in order to complete the proof, it suffices to show that the operators $M(J, W_k)$ $(k=1,2,\ldots,m)$ are Jordan block operators. In order to do this, we observe that $W_k\colon \mathfrak{H}_k\to\tilde{\mathfrak{H}}_k=\mathfrak{H}_k\oplus\ldots\oplus\mathfrak{H}_k$ is defined by the $n_k\times 1$ matrix all of whose entries are $0_{\mathfrak{H}_k}$ except the last, which is $1_{\mathfrak{H}_k}$. Hence it is clear that the operator $M(J_k, W_k)$ is the Jordan block operator of order n_k+1 on the direct sum of n_k+1 copies of $\mathfrak{H}_k$. Thus the proof is complete.

Lemma 2. *Suppose T is a nilpotent operator of order $n>1$ on a Hilbert space $\mathfrak{H}$. Then there exist Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, a nilpotent operator A of order $n-1$ in $\mathfrak{L}(\mathfrak{K}_1)$, and a bounded linear transformation $B\colon \mathfrak{K}_2\to\mathfrak{K}_1$ such that T is unitarily equivalent to the operator $M(A, B)$ in $\mathfrak{L}(\mathfrak{K}_1\oplus\mathfrak{K}_2)$ and such that $(\mathfrak{R}(A)+\mathfrak{R}(B))^-=\mathfrak{K}_1$. Furthermore, if each $\mathfrak{R}(T^k)$ is closed $(k=1,2,\ldots)$, then $\mathfrak{R}(A^k)$ is closed $(k=1,2,\ldots)$, and in this case $\mathfrak{R}(A)+\mathfrak{R}(B)=\mathfrak{K}_1$.*

Proof. Let $\mathfrak{K}_1=\mathfrak{R}(T)^-$ and $\mathfrak{K}_2=\mathfrak{H}\ominus\mathfrak{R}(T)^-$. The operator T is clearly unitarily equivalent to some operator $M(A, B)$ in $\mathfrak{L}(\mathfrak{K}_1\oplus\mathfrak{K}_2)$ where $\mathfrak{R}(A)+\mathfrak{R}(B)=\mathfrak{R}(T)$. Hence we have $(\mathfrak{R}(A)+\mathfrak{R}(B))^-=\mathfrak{K}_1$. An elementary calculation shows that A is a nilpotent operator of order $n-1$. If $\mathfrak{R}(T^k)$ is closed, $k=1,2,\ldots$, then it is clear that $\mathfrak{R}(A)+\mathfrak{R}(B)=\mathfrak{K}_1$ and it follows easily that $\mathfrak{R}([M(A, B)]^k)=\mathfrak{R}(A^{k-1})\oplus 0$ $(k=1,2,\ldots)$. Hence $\mathfrak{R}(A^k)$ is closed, $k=1,2,\ldots$, and the proof is complete.

Lemma 3. *Suppose T is a nilpotent operator on a Hilbert space $\mathfrak{H}$* [and $\mathfrak{R}(T^k)$ is closed $(k=1,2,\ldots)$]. *Then T is a quasiaffine transform of* [similar to] *a Jordan operator.*

Proof. We prove the lemma by induction on the order n of T. If $n=1$, then T is the zero operator on $\mathfrak{H}$ and hence, by definition, T is a Jordan operator. So we assume $n>1$ and that the lemma is true for all nilpotent operators of order $n-1$. According to Lemma 2, T is unitarily equivalent to an operator $M(A, B)$ in $\mathfrak{L}(\mathfrak{K}_1\oplus\mathfrak{K}_2)$ for some Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, where A is a nilpotent operator of order $n-1$ and $(\mathfrak{R}(A)+\mathfrak{R}(B))^-=\mathfrak{K}_1$ [$\mathfrak{R}(A)+\mathfrak{R}(B)=\mathfrak{K}_1$ and each $\mathfrak{R}(A^k)$ is closed]. Thus, by the induction hypothesis, there exist a Jordan operator J on a Hilbert space $\mathfrak{H}_0$ and a quasiaffinity [an invertible bounded linear transformation] $X\colon \mathfrak{K}_1\to\mathfrak{K}_0$ such that $XA=JX$. The bounded linear transformation $X\oplus 1_{\mathscr{K}_2}\colon \mathfrak{K}_1\oplus\mathfrak{K}_2\to\mathfrak{H}_0\oplus\mathfrak{K}_2$ is a quasiaffinity [is invertible] and $(X\oplus 1_{\mathscr{K}_2})M(A, B)=M(J, C)(X\oplus 1_{\mathscr{K}_2})$ where $C=XB\colon \mathfrak{K}_2\to\mathfrak{H}_0$. It is easy to verify that $(\mathfrak{R}(J)+\mathfrak{R}(C))^-=\mathfrak{H}_0$ [$\mathfrak{R}(J)+\mathfrak{R}(C)=\mathfrak{H}_0$].

We observe that $\mathfrak{R}(J)$ is closed since J is a Jordan operator. Let E be the orthogonal projection onto $\mathfrak{R}(J)$. Then, of course, $\mathfrak{R}(EC)\subset\mathfrak{R}(J)$. It follows from a theorem of R. G. Douglas ([2], Theorem 1) that there exists a bounded linear transformation Y: $\mathfrak{K}_2\to\mathfrak{H}_0$ such that $EC=JY$. The operator

$$\begin{pmatrix} 1_{\mathfrak{H}_0} & Y \\ 0 & 1_{\mathfrak{K}_2} \end{pmatrix}$$

in $\mathfrak{L}(\mathfrak{H}_0\oplus\mathfrak{K}_2)$ is invertible and

$$\begin{pmatrix} 1_{\mathfrak{H}_0} & Y \\ 0 & 1_{\mathfrak{K}_2} \end{pmatrix}\begin{pmatrix} J & C \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} J & D \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1_{\mathfrak{H}_0} & Y \\ 0 & 1_{\mathfrak{K}_2} \end{pmatrix}$$

where $D=-JY+C=-EC+C=(1_{\mathfrak{H}_0}-E)C$. A straight forward calculation shows that $\mathfrak{R}(D)^-=\mathfrak{H}_0\ominus\mathfrak{R}(J)$ $[R(D)=\mathfrak{H}_0\ominus\mathfrak{R}(J)]$.

Let $\mathfrak{K}_3=\mathfrak{K}_2\ominus\mathfrak{K}(D)$, $\mathfrak{K}_4=\mathfrak{K}(D)$, and let $D_0\colon\mathfrak{K}_3\to\mathfrak{H}_0$ be defined by $D_0x=Dx$ for each x in $\mathfrak{K}_3$. The operator $M(J,D)$ in $\mathfrak{L}(\mathfrak{H}_0\oplus\mathfrak{K}_2)$ is unitarily equivalent to the operator

$$\begin{pmatrix} J & D_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $\mathfrak{L}(\mathfrak{H}_0\oplus\mathfrak{K}_3\oplus\mathfrak{K}_4)$. So in order to complete the proof of the lemma, it suffices to show that the operator $M(J,D_0)$ in $\mathfrak{L}(\mathfrak{H}_0\oplus\mathfrak{K}_3)$ is a quasiaffine transform of [similar to] a Jordan operator. We observe that $\mathfrak{K}(D_0)=(0)$ and $\mathfrak{R}(D_0)^-=\mathfrak{H}_0\ominus\mathfrak{R}(J)$ $[\mathfrak{R}(D_0)==\mathfrak{H}_0\ominus\mathfrak{R}(J)]$. Write $D_0=VP$, the polar decomposition of D_0. It follows that V: $\mathfrak{K}_3\to\mathfrak{H}_0$ is an isometry and $\mathfrak{R}(V)=\mathfrak{R}(D_0)^-=\mathfrak{H}_0\ominus\mathfrak{R}(J)$. The operator P in $\mathfrak{L}(\mathfrak{K}_3)$ is a quasiaffinity [an invertible operator] since P is positive and $\mathfrak{K}(P)=(0)$ [and $\mathfrak{R}(P)$ is closed]. Hence the operator $1_{\mathfrak{H}_0}\oplus P$ in $\mathfrak{L}(\mathfrak{H}_0\oplus\mathfrak{K}_3)$ is a quasiaffinity [an invertible operator] and $(1_{\mathfrak{H}_0}\oplus P)M(J,D_0)=M(J,V)(1_{\mathfrak{H}_0}\oplus P)$. The operator J and the linear transformation V satisfy the hypotheses of Lemma 1. Thus the operator $M(J,V)$ is unitarily equivalent to a Jordan operator, and hence the proof is complete.

Corollary 1. *Every nilpotent operator on a Hilbert space is quasisimilar to its adjoint.*

Proof. Suppose T is a nilpotent operator. By Lemma 3, there exist a quasiaffinity X and a Jordan operator J such that $XT=JX$. Then $T^*X^*=X^*J^*$. Since every Jordan operator is unitarily equivalent to its adjoint, we have $UJ=J^*U$ where U is a unitary operator. Combining these equations, we get $(X^*UX)T=T^*(X^*UX)$. Hence T is a quasiaffine transform of T^*. The same argument applied to T^* shows that T^* is a quasiaffine transform of T. Hence T and T^* are quasisimilar.

Corollary 2. *If T is a nilpotent operator on a Hilbert space and each $\mathfrak{R}(T^k)$ is closed $(k=1,2,\ldots)$, then T is similar to its adjoint.*

Proof. By Lemma 3, there exist an invertible bounded linear transformation X and a Jordan operator J such that $XT=JX$. Now proceed as in the proof of Corollary 1 to obtain the equation $(X^* UX)T=T^*(X^* UX)$ where U is a unitary operator. Hence T and T^* are similar.

Proof of Theorem 1. Suppose T is a nilpotent operator on a Hilbert space. Then T^* is also a nilpotent operator. Thus, according to Lemma 3, there exist quasiaffinities X and Y and Jordan operators J_1 and J_2 such that $XT=J_1 X$ and $YT^*=J_2 Y$. Then $T^* X^*=X^* J_1^*$ and $TY^*=Y^* J_2^*$. Since J_1 and J_2 are Jordan operators, we have $UJ_1=J_1^* U$ and $VJ_2=J_2^* V$ where U and V are unitary operators. Combining these equations, we get $T(Y^* VYX^* U)=(Y^* VYX^* U)J_1$. Hence T and J_1 are quasisimilar.

Proof of Theorem 2. Let T be a nilpotent operator on a Hilbert space. If T is similar to a Jordan operator J, then T^k is similar to J^k for each positive integer k. It is clear that $\mathfrak{R}(J^k)$ is closed ($k=1, 2, \ldots$). Hence $\mathfrak{R}(T^k)$ is closed, $k=1, 2, \ldots$. On the other hand if $\mathfrak{R}(T^k)$ is closed ($k=1, 2, \ldots$), then we can conclude from Lemma 3 that T is similar to a Jordan operator.

Foiaş and Pearcy [3] proved that every nilpotent operator acting on a separable Hilbert space is quasisimilar to a compact operator. Below we give a different proof of this theorem based on the following lemma.

Lemma 4. *If T is a nilpotent operator on a separable Hilbert space $\mathfrak{H}$, then there exist a compact quasiaffinity Z and a compact operator K in $\mathfrak{L}(\mathfrak{H})$ such that $ZT=KZ$.*

Proof. We prove the lemma by induction on the order n of T. If $n=1$, then T is the zero operator on $\mathfrak{H}$ and the result is obvious. So we assume $n>1$ and that the lemma is true for all nilpotent operators of order $n-1$ acting on a separable Hilbert space. According to Lemma 2, the operator T is unitarily equivalent to an operator $M(A, B)$ in $\mathfrak{L}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)$ for some separable Hilbert spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, where A is a nilpotent operator of order $n-1$ in $\mathfrak{L}(\mathfrak{K}_1)$. Thus by the induction hypothesis, there exist a compact quasiaffinity Z_0 and a compact operator K_0 in $\mathfrak{L}(\mathfrak{K}_1)$ such that $Z_0 A = K_0 Z_0$. Write $Z_0 B=UP$, the polar decomposition of $Z_0 B$. The operator P in $\mathfrak{L}(\mathfrak{K}_2)$ is positive and compact. Hence $P^{1/2}$ is compact. Let $\tilde{P}$ be any compact quasiaffinity in $\mathfrak{L}(\mathfrak{K}(P^{1/2}))$. We define a compact quasiaffinity P_0 on $\mathfrak{K}_2$ by setting $P_0 x=\tilde{P}x$ for each x in $\mathfrak{K}(P^{1/2})$ and $P_0 x=P^{1/2}x$ for each x in $\mathfrak{K}_2 \ominus \mathfrak{K}(P^{1/2})$. Clearly $P=P^{1/2} P_0$. The operator $Z_0 \oplus P_0$ is a compact quasiaffinity and the operator $M(K_0, UP^{1/2})$ is compact. An easy calculation shows that $(Z_0 \oplus P_0) M(A, B)=M(K_0, UP^{1/2})(Z_0 \oplus P_0)$, and hence the proof is complete.

Theorem 3. *Every nilpotent operator on a separable Hilbert space is quasisimilar to a compact operator.*

Proof. Suppose T is a nilpotent operator on a separable Hilbert space. According to Lemma 4, there exist a (compact) quasiaffinity Z and a compact operator K such that $ZT=KZ$. Then $T^*Z^*=Z^*K^*$. The operator K is necessarily nilpotent. Thus, by applying Corollary 1 to T and K, we can obtain quasiaffinities X and Y such that $TX=XT^*$ and $YK=K^*Y$. Combining these equations, we get $T(XZ^*Y)=$ $=(XZ^*Y)K$. Hence T and K are quasisimilar.

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